# $\epsilon$-Approximate Selection

Based on

Approximate Medians and other Quantiles in One Pass and with Limited Memory

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# Background: Median

Given $(a_1, \ldots, a_N)$ from an ordered set:

## Median Problem:

- Find element that evenly splits elements.
- Useful statistic.
- Applies to non-numerical data.

## Current activity:

- Focused on minimal number of comparisons.

# Generalization of Median

Two forms of the same problem:

1. Quantile Problem:

   Find element at $\lceil \phi N \rceil$.

2. Selection Problem:

   Find element at location $k \leq N$.

Used for:

- Order Statistics.
- Equal depth histograms.
- Other?

Solutions:

- $O(N)$ probabilistic: (Modified quicksort). By Tony Hoare, 1961
- $O(N)$ deterministic. By Blum, Floyd, Pratt, Rivest, and Tarjan, 1973

# <u>The Problem</u>

**Def. 1 ( $\epsilon$-Approximate Selection )**

*Given* $P \leq N$*,*

*Return:* $V$ *within* $\lceil \epsilon N \rceil$ *in the sorted order.*

**One-Pass $\epsilon$-Approximate Selection Problem**

We are given

- $\epsilon$, the approximation accuracy,
- $N$, the size of the data sequence,
- $S := (a_1, a_2, \ldots, a_N)$.

Allowed one pass over the data

to form a summary data object,

for $\epsilon$-approximating selection queries.

The goal is use a "small amount of memory" while forming this summary data structure.

# A Solution?

Can solve problem exactly

- using $O(N)$ space,
- in $N \ln(N)$ time,
- for $O(1)$ queries.

Just

sorting the data,

placing the results in an array.

For $\epsilon$-approximation:

Select evenly spaced elements

separated by $\epsilon N$

yields summary data object of size $1/\epsilon$

# A Solution

Problem: $N \gg$ available memory

**Goal:** We will cover a method that yields an $\epsilon$-approximation using $O(\epsilon^{-1}[\ln(\epsilon N)]^2)$ space.

# Construction Operators

Build summary structure on incoming data.

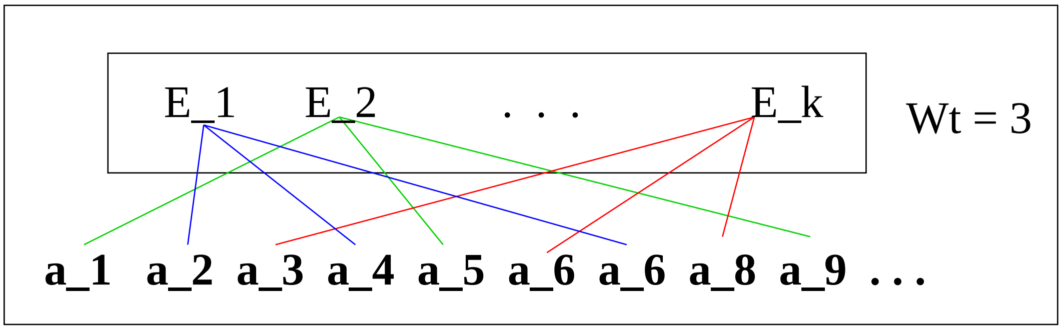
They manage

**Buffers:** $B$ buffers, each of size $K$.

Active buffer $Y$ contains $K$ elements.

Each element in $Y$ "summarizes" or

represent $Y$.wt elements from sequence.




**Toggle:** This has a technical purpose.

# $\underline{\texttt{New()}}$

- fills empty buffer $Y$ from sequence.
- sorts the elements,
- assign $Y$ a weight of one: $Y$.wt $:= 1$,
- sets $Y$.offset $:= 1$,
- $Y$.used $:=$ $\texttt{True}$.

$$
\begin{array}{cccccccccc}
 & a & q & c & w & n & p & b & s & \ldots \\
 & & & \downarrow & & & & & & \\
| & a & q & c & w & n & | & & & \\
 & & & \downarrow & & & & & & \\
| & a & c & n & q & w & | & & &
\end{array}
$$

If necessary pad $N$ to multiple of $K$, with

$-\infty$ in the first buffer.

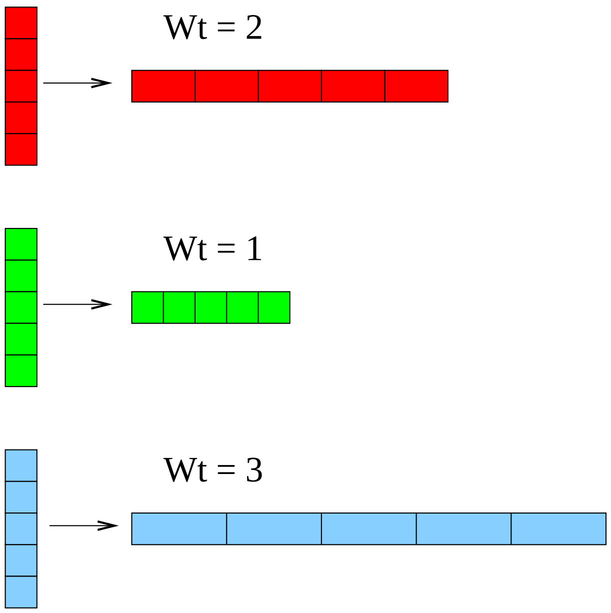
$+\infty$ in the last buffer

# Collapse()-1

$Y_1, \ldots, Y_k$, $k \geq 2$ full buffers $\rightarrow$ empty $Y_0$:

Virtually:

**1.** Replicate elements of $Y_i$ by $Y_i$.wt,



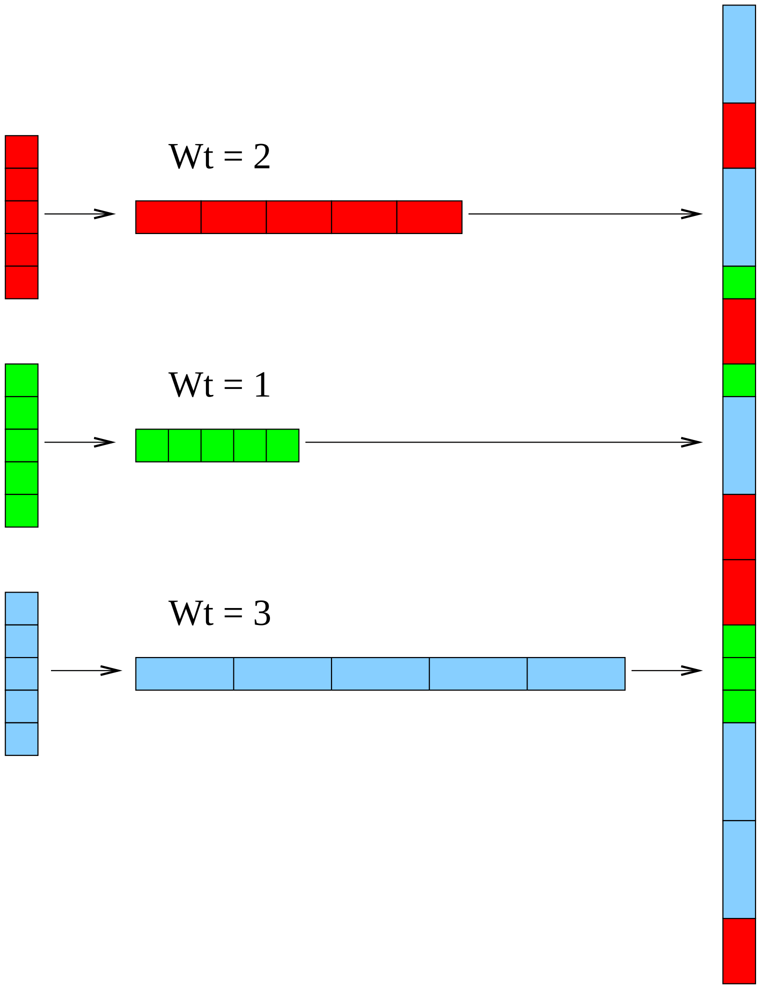
# Collapse()-2

$Y_1, \ldots, Y_k$, $k \geq 2$ full buffers $\rightarrow$ empty $Y_0$:

Virtually:

**1.** Replicate elements of $Y_i$ by $Y_i$.wt,

**2.** Merge sorted buffers.

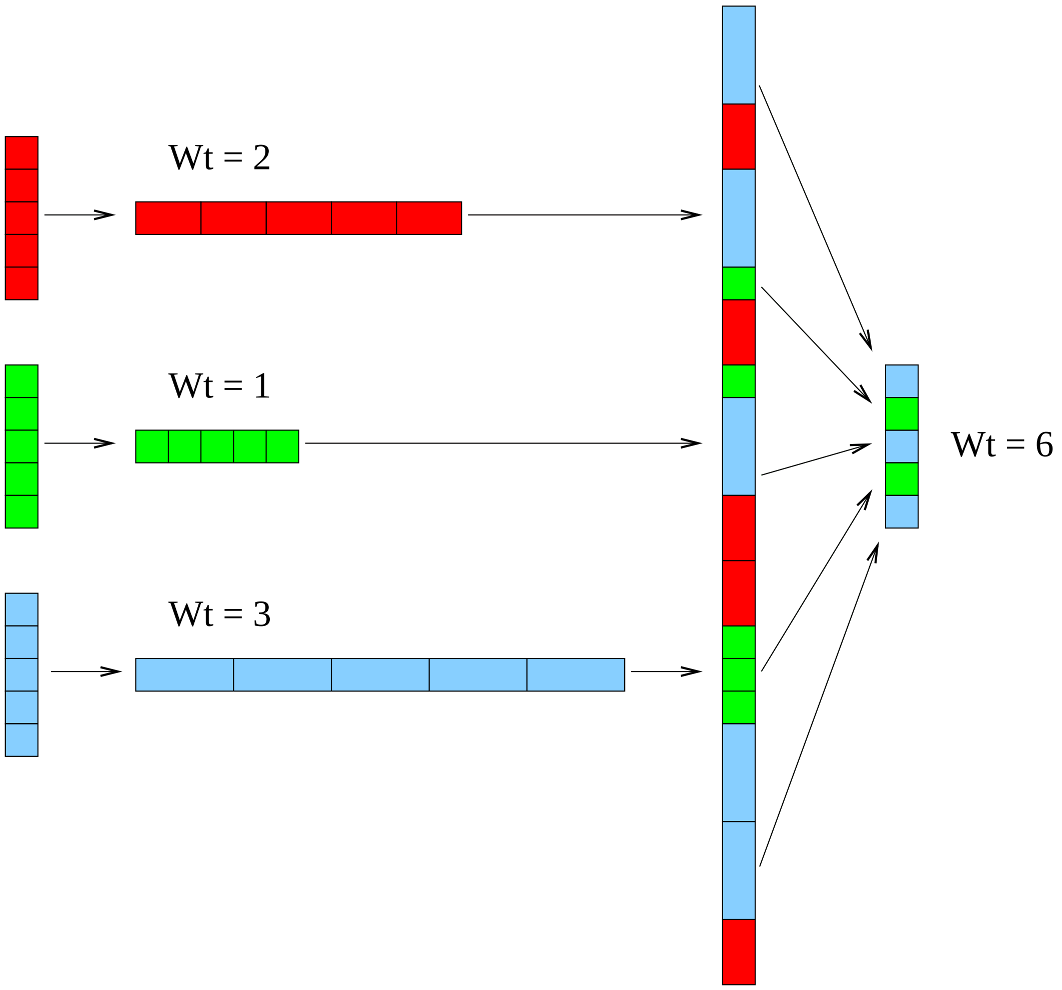


# Collapse()-3

$Y_1, \ldots, Y_k$, $k \geq 2$ full buffers $\rightarrow$ empty $Y_0$:

Virtually:

1. Replicate elements of $Y_i$ by $Y_i$.wt,
2. Merge sorted buffers.
3. Select $K$ evenly spaced elements



# Collapse()-Weighted Merge

1. $Y_0.\mathsf{wt} := Y_1.\mathsf{wt} + \cdots + Y_k.\mathsf{wt}$.

2. $Y_0.\mathsf{offset} :=$

$$\begin{cases} \dfrac{Y_0.\mathsf{wt}+1}{2} & \text{if } Y_0.\mathsf{wt} \text{ is odd,} \\ \dfrac{Y_0.\mathsf{wt}}{2} + \texttt{Toggle} & \text{if } Y_0.\mathsf{wt} \text{ is even.} \\ & \texttt{Toggle := 1 - Toggle}. \end{cases}$$

3. Select $K$ elements located:
   $Y_0.\mathsf{offset} + jY_0.\mathsf{wt}$, for $j = 0, \ldots, (K-1)$.

**Initialize:**

priority queue with
$(Y_1[1], 1), \ldots, (Y_k[1], k)$;

virtual_element_count := 0;

**While** priority queue not empty:

Dequeue least element $(e, i)$;

Enqueue next from $Y_i$;

virtual_element_count $+=$ $Y_i.\mathsf{wt}$;

If passes selection point,
  add $e$ to $Y_0$;

# `Collapse()`

$Y_1, \ldots, Y_k$, $k \geq 2$ full buffers $\rightarrow$ empty $Y_0$:

Last, set the status of the buffers:

**4.** $Y_0$.used := `True`.

**5.** Mark $Y_1$.used$, \ldots, Y_k$.used to `False`.

Note, could collapse into one of the old buffers.

# Completion/Query Operators

Finalize(): Convert buffers to query format.

Merge sort active buffers.

Convert ordered sequence of `(value, weight)`,

to ordered array of `(value, offset)`.

$$(a, 8), (b, 2), (c, 7), (d, 2), (e, 8), \cdots$$

$$\downarrow$$

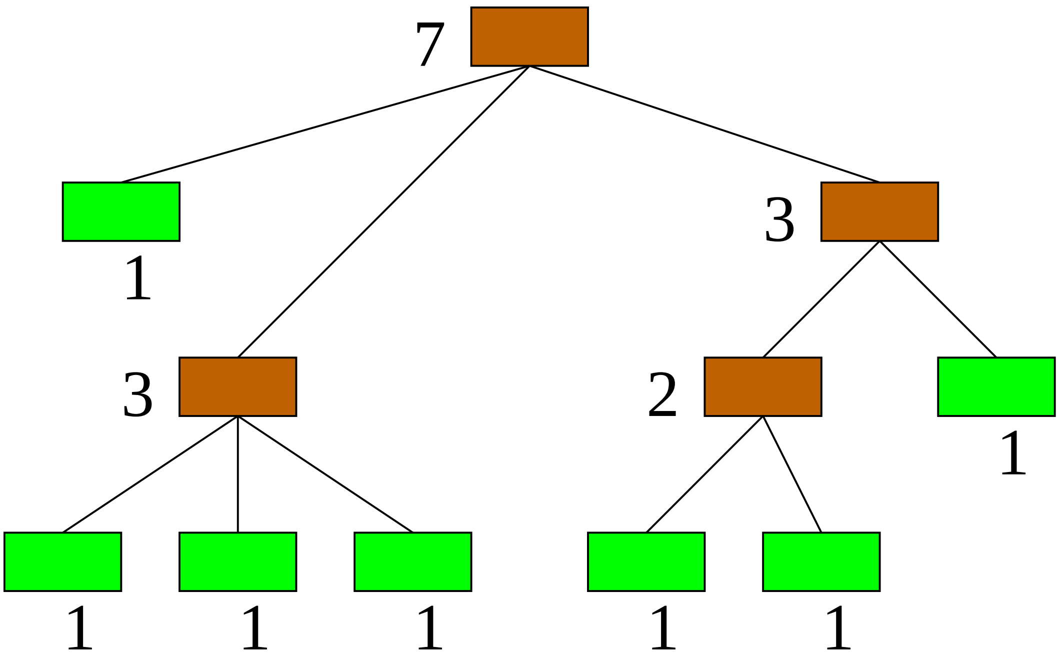
$$(a, 8), (b, 10), (c, 17), (d, 19), (e, 27), \cdots$$

Select(): Given $P \leq N$,
use bisection to seek candidate $V$.

# History Forest

- Various algorithms built from operators.
- Differ only in use of New() and Collapse().
- Policy reflected in the evolution of buffers.
- Can view buffer evolution as forest.
  - New() creates the leaves.
  - Collapse() produces internal nodes.

A Tree in a History Forest:



# History Forest-2

Note

- Roots are active buffers.
- Internal nodes have weight equal to number of leaf decedents.

**Lemma 2** *Mirror of a history tree is a history tree.*

**P:** Let $\{a_1, \ldots, a_N\} \in (O, \leq)$, we

- Invert order of sequence: $(i \mapsto N + 1 - i)$,
- Use the reverse order on $O$,
- Invert uses of $-\infty, +\infty$.
- Possibly start `toggle` with different value.

The $P$-th selection of the original corresponds to the $N+1-P$-th selection in the mirror. $\square$

# Some Analysis

Given $P \leq N$, Select() returns $V$.

How far is $V$ from $P$?

**Def. 3** *We define some variables:*

$P_V$ *Real location of $V$ in sorted sequence.*

$C$ *The number of Collapse() operations, or internal nodes in history forest.*

$W_{max}$ *Weight of heaviest root (active buffer).*

$W_{\Sigma}$ *Sum of the weights of internal nodes.*

# Some Analysis-2

Assume no repetitions in data (worst case).

We will show

$$P_V > P - \left(\frac{W_\Sigma - C - 1}{2} + W_{max}\right).$$

Mirror tree has same values for:
$C$, $W_{max}$, and $W_\Sigma$.

Symmetry will imply

$$P_V < P + \left(\frac{W_\Sigma - C - 1}{2} + W_{max}\right).$$

Establishing localization bound

$$|P_V - P| < \frac{W_\Sigma - C - 1}{2} + W_{max}.$$

# Special Elements

**Def. 4 ( Special Elements )**
*For any root $X$,*

$$X.\text{special} := \{\, v \in X \mid v < V \,\}.$$

$\equiv$ *elements in final array less than $V$.*

*Once* $Y.\text{special}$ *determined, let*
$Y.\text{max\_special} :=$

$$\begin{cases} \max Y.\text{special} & \textit{if } Y.\text{special} \neq \emptyset, \\ -\infty & \textit{otherwise.} \end{cases}$$

*If $Y$ has children, proceed recursively*

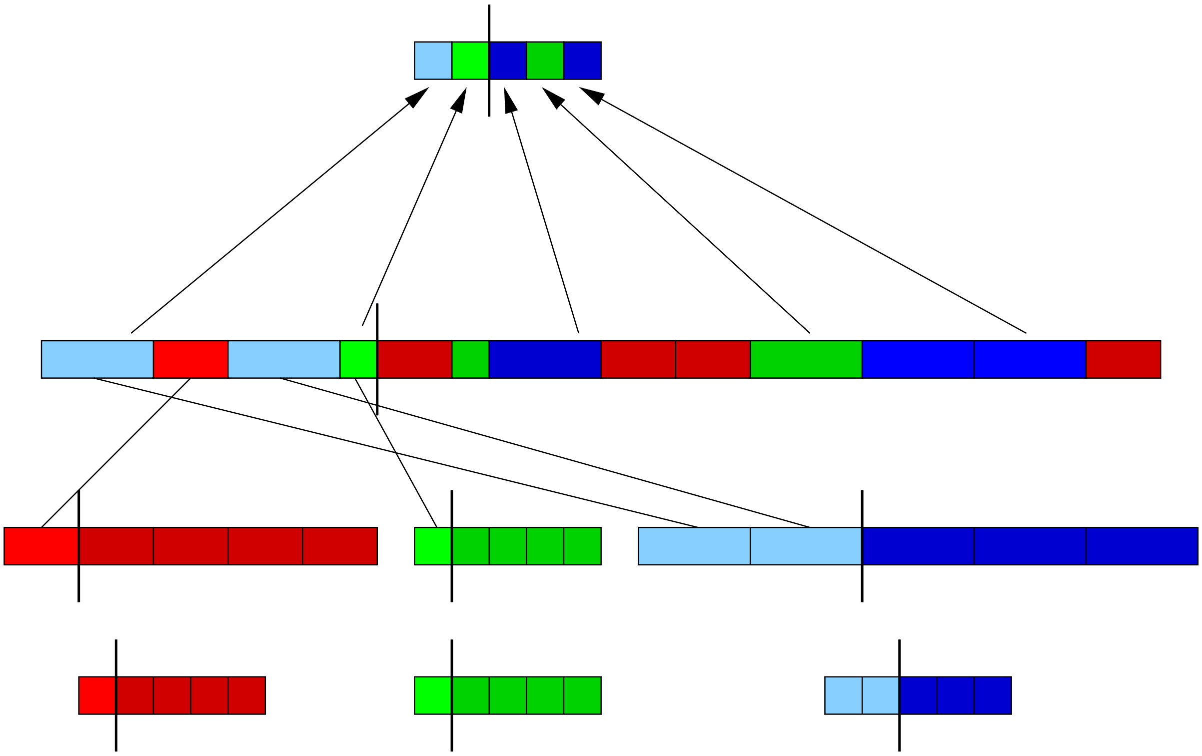
*For* $Z \in Y.\text{children}$

$$Z.\text{special} := \{\, v \in Z \mid v \leq Y.\text{max\_special} \,\}.$$

# Special Elements

For $Z \in Y$.children

$$Z.\text{special} := \{ v \in Z \mid v \leq Y.\text{max\_special} \}.$$



# Weighted Count of Specials

**Def. 5** *We set*

$$Y.\text{wt\_special} := Y.\text{wt} * |Y.\text{special}|$$

*weighted count of special elements in $Y$.*

*Let*

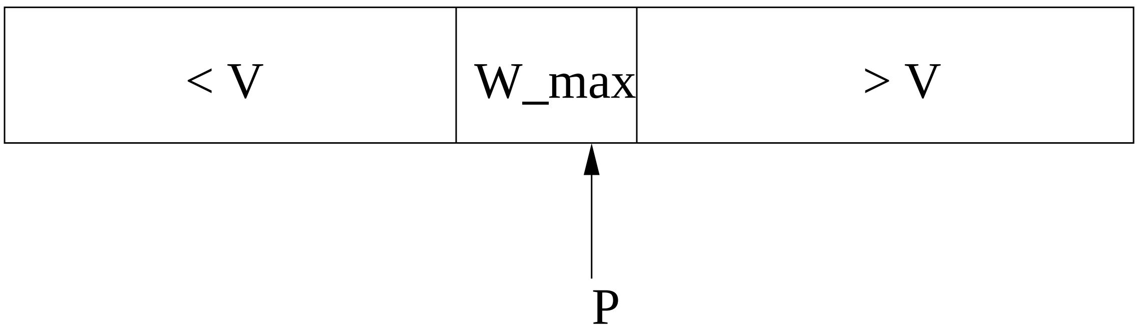
$$Roots.\text{wt\_special} := \sum_{X \textit{ is a root}} X.\text{wt\_special}$$

*weighted count of special elements in roots.*

**Lemma 6**

$$P \leq Roots.\text{wt\_special} + W_{max}.$$

**P:** In final array, element in position $P$ has multiplicity of 1 to $W_{max}$.




□

# Weighted Count under Recursion

**Lemma 7** *If $Y$ has children, then*

$$\sum_{Z \in Y.\text{children}} Z.\text{wt\_special} \geq Y.\text{wt\_special} - (Y.\text{wt} - Y.\text{offset}).$$

**P:** **Case $Y$.special $= \emptyset$:** LHS $\geq 0 \geq$ RHS.

**Case $Y$.special $\neq \emptyset$:**

- $Y$.wt_special := $|Y.\text{special}| * Y.\text{wt}$.
- $Y$.max_special located at $|Y.\text{special}|$.
- Picked from location

$$(|Y.\text{special}| - 1) * Y.\text{wt} + Y.\text{offset}$$

$$= |Y.\text{special}| * Y.\text{wt} - (Y.\text{wt} - Y.\text{offset})$$

$$= Y.\text{wt\_special} - (Y.\text{wt} - Y.\text{offset}).$$

These are exactly the elements in the children that will get distinguished as special. So we have equality. $\square$

# Accumulated Effects

Proceeding from parent to children, reduces weight count of special elements by at most

$$Y.\mathrm{wt} - Y.\mathrm{offset}.$$

So reduction from roots to leaves: bounded by

$$\sum_{Y \in \mathrm{internal}} (Y.\mathrm{wt} - Y.\mathrm{offset})$$

$$= W_\Sigma - \sum_{Y \in \mathrm{internal}} Y.\mathrm{offset}.$$

As the weight of the leaf nodes is one, we have shown:

**Lemma 8** *Let* $Leaves.\mathrm{special}$ *be the number of special elements in the leaves. Then*

$$\begin{aligned} & Leaves.\mathrm{special} \\ \geq\ & Roots.\mathrm{wt\_special} - W_\Sigma + \sum_{Y \in \textit{internal}} Y.\mathrm{offset}\,. \end{aligned}$$

$\square$

# Accumulation of Offsets

**Lemma 9** *The sum of the offsets of the collapse nodes is at least*

$$\frac{W_\Sigma + C - 1}{2}.$$

**P:** Let

$C_o$: number of internal nodes with odd weight,

$C_e^0$: number of nodes of even weight,

$C_e^0$ even weight with $Y$.offset $= Y$.wt$/2$,

$C_e^1$ even weight with $Y$.offset $= 1 + Y$.wt$/2$.

Then $C = C_o + C_e = C_o + C_e^0 + C_e^1$.

The sum of all the offsets is

$$\begin{aligned}\frac{W_\Sigma + C_o}{2} + C_e^1 &= \frac{W_\Sigma + C_o + 2C_e^1}{2} \\ &= \frac{W_\Sigma + C_o + C_e - \mathrm{mod}_2(C_e)}{2} \\ &= \frac{W_\Sigma + C - \mathrm{mod}_2(C_e)}{2}.\end{aligned}$$

□

# Lower Bound on $P_V$

So

$$
\begin{aligned}
& P_V \\
> \; & Leaves.\text{special} \\
\geq \; & Roots.\text{wt\_special} - W_\Sigma + \frac{W_\Sigma + C - 1}{2} \\
= \; & Roots.\text{wt\_special} - \frac{W_\Sigma - C + 1}{2}.
\end{aligned}
$$

**Lemma 10** *Let $P_V$ be the position of $V$ in the sorted list. Then*

$$P_V > P - \left(\frac{W_\Sigma - C + 1}{2} + W_{max}\right).$$

**P:** We have determined that

- $Roots.\text{wt\_special} - \dfrac{W_\Sigma - C + 1}{2} < P_V.$
- $P \leq Roots.\text{wt\_special} + W_{max}$,

  or equivalently
  $$P - \left(\frac{W_\Sigma - C + 1}{2} + W_{max}\right) \leq Roots.\text{wt\_special} - \frac{W_\Sigma - C + 1}{2}.$$

Transitivity now show the result. $\square$

# Uniformity of Variance

**Prop 11 ( Uniform Bound on Variance )**
*No element in the final array is more than*

$$\frac{W_{\Sigma} - C + 1}{2} + W_{max}$$

*from closest location in sorted input.*

**P:** The $P$-th element was an arbitrary.
So each element in the expanded list,
it isn't "too far away" from it's real location.

$\square$

# Basic Constraints

**Lemma 12** *The maximum number of children of any node is bounded by $B$, the number of buffers used.* $\square$

**Goal 13** We want to minimize $BK$ subject to two feasibility constraints:

1. Responses to queries are $\epsilon$-approximations:
$$\frac{W_\Sigma - C - 1}{2} + W_{max} \leq \epsilon N.$$
Depends only on forest shape.

2. History forest holds input of size $N$:
$$KL \geq N,$$
where $L$ is the number of leaf nodes.
Pick smallest such $K$.

# An Algorithm

Associate attribute of level to buffers by:

**2 or more unused buffer:**

- Fill an empty buffer
- place as an isolated node at level 0.

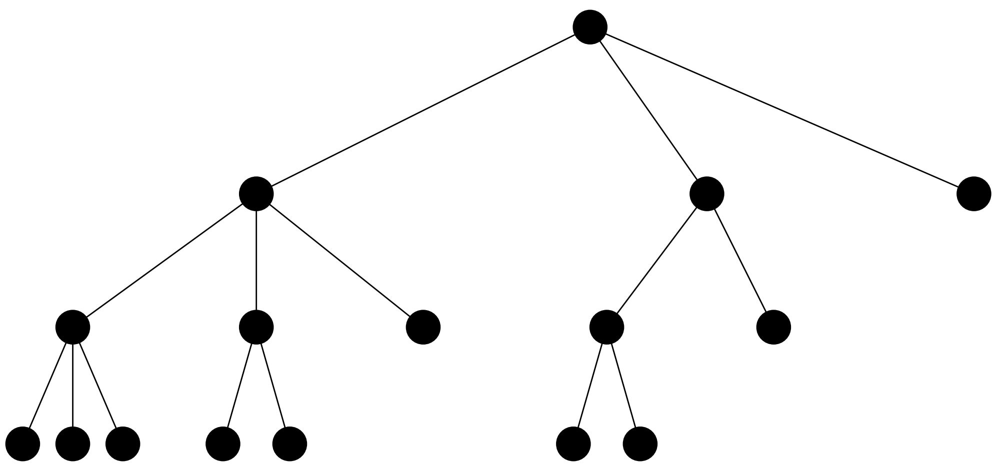
**1 unused buffer:**

- Fill the empty buffer,
- place as an isolated node at the level of the current lowest level buffer.

**0 unused buffer:**

- Collapse buffers at lowest level to buffer,
- assign at level one higher.

Example history tree for $B = 3$, $H = 3$:



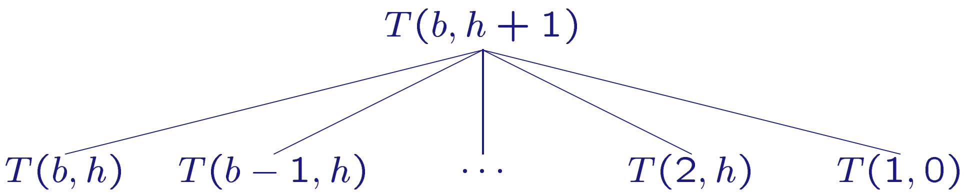
# Fractal Nature

$T(b, h)$: tree on $b$ buffers and height of $h$.

We examine individual trees in the forest:

Note

- Recursive structure when $b \geq 2$ and $h \geq 1$:




- for $b > 2$ and $h > 0$,
  if we remove first (largest) child from $T(b, h)$,
  we get $T(b-1, h)$.

- for $b > 2$ and $h > 1$,
  if we remove the bottom level of $T(b, h)$
  we get $T(b, h-1)$.

# Stats for Single Tree

**Def. 14** *Let $T(b,h)$ be the tree on $b$ buffers of height $h$. Then*

*$I(b,h)$ is the number of internal nodes.*

*$L(b,h)$ denotes the number of leaves.*

*$W_\Sigma(b,h)$ denote the sum of the weights of all the internal nodes.*

**Prop 15** *For $b \geq 2$ and $h \geq 1$,*

$$I(b,h) = \binom{b+h-2}{h-1}.$$

$$L(b,h) = \binom{b+h-1}{h}.$$

$$W_\Sigma(b,h) = h\binom{b+h-1}{h} - \binom{b+h-2}{h-2}.$$

**P:** Repeated applications of the upper summation formula.

See lemmas in appendix for the gory deals... $\square$

# Memory Size Bound

Best foot forward:
Examine constraints just before collapse into a single node: pre-$T(B,H)$

$$\epsilon N \geq \frac{W_\Sigma - C - 1}{2} + W_{max} \approx \frac{W_\Sigma}{2}$$

$$\approx \frac{1}{2} H \binom{B+H-1}{H}.$$

From $KL \geq N$ we get

$$K\,L(B,H) \geq N \quad \equiv \quad K\binom{B+H-1}{H} \geq N.$$

To establish upper bound memory usage, set $H = B$ and use equalities.

From $\epsilon N \approx \frac{1}{2} B \binom{2B-1}{B}$, we get $B \in O(\ln(\epsilon N))$.

From $N \approx K\binom{2B-1}{B} \approx K\frac{2\epsilon N}{B}$, or $K \approx \frac{B}{2\epsilon}$, we get $K \in O(\epsilon^{-1} \ln(\epsilon N))$.

Hence $KB \in O(\epsilon^{-1}[\ln(\epsilon N)]^2)$.

# Summary

Algorithm has upper bound on

Space usage:

$$KB \in O(\epsilon^{-1}[\ln(\epsilon N)]^2).$$

Query response time.

$$\ln(KB) = O(\ln(\epsilon^{-1}[\ln(\epsilon N)]^2)).$$

Further developments:

Component of probabilistic version.

Component of an online version.

# Appendix

# Counting Internal Nodes

**Lemma 16** *For $b \geq 2$ and $h \geq 1$,*

$$I(b,h) = \binom{b+h-2}{h-1}.$$

**P:** We show by induction on the height. Clear for $I(b,1) = 1 = \binom{b+1-2}{0}$.

Suppose it holds for $h \geq 1$, then

$$\begin{aligned} I(b,h+1) &= \sum_{k=2}^{b} I(k,h) + 1 \\ &= \sum_{k=2}^{b} \binom{k+h-2}{h-1} + 1 \\ &= \sum_{k=0}^{b-1} \binom{(k+1)+h-2}{h-1} \\ &= \binom{b+(h+1)-2}{(h+1)-1}. \end{aligned}$$

$\square$

# Counting Leaf Nodes

**Lemma 17** *For $b \geq 2$ and $h \geq 1$,*

$$L(b,h) = \binom{b+h-1}{h}.$$

**P:** We show by induction on the height. This is clear for $L(b,1) = b = \binom{b+1-1}{1}$.

Suppose it holds for $h \geq 1$, then

$$\begin{aligned} L(b,h+1) &= \sum_{k=2}^{b} L(k,h) + 1 \\ &= \sum_{k=2}^{b} \binom{k+h-1}{h} + 1 \\ &= \sum_{k=0}^{b-1} \binom{k+h}{h} \\ &= \binom{b+h}{h+1} \\ &= \binom{b+(h+1)-1}{h+1}. \end{aligned}$$

$\square$

# $W_{max}$

**Cor 18** *For $h \geq 2$, the weight of the maximum child in $T(b,h)$ is*

$$W_{max} = T(b, h-1) = \binom{b+h-2}{h-1}.$$

**P:** The biggest child of $T(b,h)$ is $T(b,h-1)$, so the number of leaves is

$$\binom{b+(h-1)-1}{h-1} = \binom{b+h-2}{h-1}.$$

$\square$

# Weighing All Internal Nodes

**Lemma 19** *For $b \geq 2$ and $h \geq 1$,*

$$W_\Sigma(b,h) = h\binom{b+h-1}{h} - \binom{b+h-2}{h-2}.$$

**P:** We show by induction on the height. This is clear for $W_\Sigma(b,1) = b = 1\binom{b+1-1}{1} - 0$. Suppose it holds for $h \geq 1$, then

$$
\begin{aligned}
& W_\Sigma(b,h+1) \\
&= \sum_{k=2}^{b} W_\Sigma(k,h) + L(b,h+1) \\
&= \sum_{k=2}^{b} h\binom{k+h-1}{h} - \sum_{k=2}^{b}\binom{k+h-2}{h-2} + \binom{b+(h+1)-1}{h+1} \\
&= h\sum_{k=0}^{b-1}\binom{k+h}{h} - h - \sum_{k=2}^{b}\binom{k+h-2}{h-2} + \binom{b+h}{h+1} \\
&= (h+1)\binom{b+h}{h+1} - \sum_{k=2}^{b}\binom{k+h-2}{h-2} - (h-1) - 1 \\
&= (h+1)\binom{b+h}{h+1} - \sum_{k=0}^{b}\binom{k+h-2}{h-2} \\
&= (h+1)\binom{b+(h+1)-1}{h+1} - \binom{b+(h+1)-2}{(h+1)-2}.
\end{aligned}
$$

□

# Constraints

Best foot forward:

Examine constraints just before collapse into a single node: pre-$T(B,H)$

1.
$$\begin{aligned}\epsilon N &\geq \frac{W_\Sigma - C - 1}{2} + W_{max}\\ &= \frac{(W_\Sigma(B,H) - L(B,H)) - (I(B,H) - 1) - 1}{2} + L(B,H-1).\end{aligned}$$

Which becomes

$$\begin{aligned}2\epsilon N &\geq W_\Sigma(B,H) - L(B,H) - I(B,H) + 2L(B,H-1)\\ &= \left[H\binom{B+H-1}{H} - \binom{B+H-2}{H-2}\right]\\ &\quad -\binom{B+H-1}{H} - \binom{B+H-2}{H-1} + 2\binom{B+H-2}{H-1}\\ &= (H-1)\binom{B+H-1}{H} - \binom{B+H-2}{H-2} + \binom{B+H-2}{H-1}.\end{aligned}$$

This first constraint places restrictions on the nature of the history tree, and is independent of size of the buffers.

2. From $KL \geq N$ we get

$$K\,L(B,H) \geq N \quad \equiv \quad K\binom{B+H-1}{H} \geq N.$$

# Bound on $B$

To reduce the analysis complexity,
for upper bound on best selection for $B$
we make the assumption that $B = H$.
Substitute and reduce:

$$\begin{aligned}
2\epsilon N &\geq (B-1)\binom{2B-1}{B} - \binom{2B-2}{B-2} + \binom{2B-2}{B-1} \\
&= B\binom{2B-1}{B} - \binom{2B-1}{B} - \binom{2B-2}{B-2} + \binom{2B-2}{B-1} \\
&= B\binom{2B-1}{B} - \binom{2B-2}{B} - \binom{2B-2}{B-1} - \binom{2B-2}{B-2} + \binom{2B-2}{B-1} \\
&= B\binom{2B-1}{B} - \binom{2B-2}{B} - \binom{2B-2}{B-2} \\
&= B\binom{2B-1}{B} - 2\binom{2B-2}{B}.
\end{aligned}$$

If we select largest $B$ satisfying

$$2\epsilon N \geq B\binom{2B-1}{B},$$

then it satisfies the previous inequality.

FIXME:

For this selection of $B$ we have

$B \in O(\ln(\epsilon N))$.

# Bound on $K$

From $B = H$, our previous constraint reduces to

$$K\binom{2B-1}{B} \geq N,$$

for which we want the smallest such $K$.

The selection of $B$ satisfies

$$2\epsilon N = B\binom{2B-1}{B} \qquad \text{or} \qquad \frac{2\epsilon N}{B} = \binom{2B-1}{B},$$

Substitution yields for this choice of $B$

$$K\frac{2\epsilon N}{B} \geq N \qquad \text{or} \qquad K = \frac{B}{2\epsilon}$$

$$\text{which implies} \qquad K \in O(\epsilon^{-1}\ln(\epsilon N)).$$

Finally, we combine these two upper bounds in

$$\begin{aligned} KB &\in O(\epsilon^{-1}\ln(\epsilon N))O(\ln(\epsilon N)) \\ &= O(\epsilon^{-1}(\ln(\epsilon N))^2). \end{aligned}$$